New Expressions for Discrete Painlevé Equations

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1 Introduction

Discrete Painlevé equations are studied from various points of view as integrable systems [2], [7]. They are discrete equations which are reduced to the Painlevé differential equations in a suitable limiting process, and moreover, which pass the singularity confinement test. Passing this test can be thought of as a difference version of the Painlevé property. The Painlevé differential equations were derived as second order ordinary differential equations whose solutions have no movable singularities other than poles. This property is called the Painlevé property. The singularity confinement test has been proposed by Grammaticos *et al.* as a criterion for the integrability of discrete dynamical systems [1]. It demands that singularities depending on particular initial values should disappear after a finite number of iteration steps, in which case the information about the initial values ought to be recovered.

H. Sakai constructed all difference Painlevé equations from the point of view as algebraic geometry [8]. Surfaces obtained by successive blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied by means of connections between Weyl groups and groups of Cremona isometries of the Picard group of surfaces. The Picard group of a rational surface X is the group of isomorphism classes of invertible sheaves on X and is isomorphic to the group of linear equivalence classes of divisors on X. A Cremona isometry is an isomorphism of the Picard group that i) preserves the intersection number of any pair of divisors, ii) preserves the canonical divisor $\mathcal{K}_X$ and iii) leaves the set of effective classes of divisors invariant. In the case where eight points in general positions are blown up in $\mathbb{P}^1 \times \mathbb{P}^1$, the group of Cremona isometries of X is isomorphic to an extension of the Weyl group of type $E_8^{(1)}$. Birational mappings on $\mathbb{P}^1 \times \mathbb{P}^1$ are obtained by subsequent blow downs. Discrete Painlevé equations are recovered as the birational mappings corresponding to translations of affine Weyl groups.

Discrete Painlevé equations were classified on the basis of the types of rational surfaces connected to extended affine Weyl groups and some new equations were discovered in the process. See Table 1.

Table 1: Classification of generalized Halphen surfaces with $\dim|-\mathcal{K}_X| = 0$

type	surface (symmetry)
Elliptic type	$A_0^{(1)}(E_8^{(1)})$
Multiplicative type	$A_0^{(1)*}(E_8^{(1)})$ $A_1^{(1)}(E_7^{(1)})$ $A_2^{(1)}(E_6^{(1)})$ $A_3^{(1)}(D_5^{(1)})$
	$A_4^{(1)}(A_4^{(1)})$ $A_5^{(1)}((A_2+A_1)^{(1)})$ $A_6^{(1)}((A_1+\underset{\lvert\alpha\rvert^2=14}{A_1})^{(1)})$
	$A_7^{(1)}(\underset{\lvert\alpha\rvert^2=8}{A_1^{(1)}})$ $A_7^{(1)\prime}(A_1^{(1)})$ $A_8^{(1)}(A_0^{(1)})$
Additive type	$A_0^{(1)**}(E_8^{(1)})$ $A_1^{(1)*}(E_7^{(1)})$ $A_2^{(1)*}(E_6^{(1)})$
	$D_4^{(1)}(D_4^{(1)})$ $D_5^{(1)}(A_3^{(1)})$ $D_6^{(1)}((2A_1)^{(1)})$
	$D_7^{(1)}(\underset{\lvert\alpha\rvert^2=4}{A_1^{(1)}})$ $D_8^{(1)}(A_0^{(1)})$
	$E_6^{(1)}(A_2^{(1)})$ $E_7^{(1)}(A_1^{(1)})$ $E_8^{(1)}(A_0^{(1)})$

These equations are organized in a degeneration pattern obtained through coalescence. The $A_0^{(1)}$-surface discrete Painlevé equation ($dP(A_0^{(1)})$) is the most generic one among these, because the equation has Weyl group symmetry of type $E_8^{(1)}$ and any of the other discrete Painlevé equations can be obtained from this equation by limiting procedure. The form of this equation is however very complicated as its coefficients are described in terms of elliptic functions [5], [6].

Kajiwara *et al.* presented a τ function formalism for the elliptic discrete Painlevé equation [4]. They gave an explicit form of $dP(A_0^{(1)})$ based on this formalism.

In this paper a new representation of $dP(A_0^{(1)})$ based on total transforms is presented. Kajiwara *et al.*'s form is described in terms of coordinates in $\mathbb{P}^2$, while our form is described by means of $\mathbb{P}^1 \times \mathbb{P}^1$. Both descriptions are related by a bitational transform [5]. However, as in our construction all eight points (that are blown up in $\mathbb{P}^1 \times \mathbb{P}^1$) appear in a symmetric way, the symmetries of $dP(A_0^{(1)})$ are immediately apparent.

In Section 2, we present the representation of $dP(A_0^{(1)})$. In Section 3–8, we present expressions for other discrete Painlevé equations, obtained in a similar way. This is made possible by the fact that the limit process is a projective transformation of dependent variables.

Consequently, it can be seen that knowledge of the blow-up of eight points in $\mathbb{P}^1 \times \mathbb{P}^1$ and of a trivial solution on a curve passing through these points allows one to construct a discrete Painlevé equation in explicit form.

2 $A_0^{(1)}$-surface

In this section, we give a new representation of the $A_0^{(1)}$-surface discrete Painlevé equation ($dP(A_0^{(1)})$). First we present some useful notation.

We construct the $A_0^{(1)}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points p_i ($i = 1, \ldots, 8$). There exists an elliptic curve that passes through generic eight points. We parametrize these eight points and the curve as follows:

$$\begin{aligned}(f_1 g_0 + f_0 g_1 + \wp(2t) f_0 g_0)(4\wp(2t) f_1 g_1 - g_3 f_0 g_0)& \\ = \left(f_1 g_1 + \wp(2t)(f_1 g_0 + f_0 g_1) + \frac{g_2}{4} f_0 g_0\right)^2&,\end{aligned} \tag{2.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = (1 : \wp(b_i + t),\ 1 : \wp(t - b_i)) \quad (i = 1, \ldots, 8). \tag{2.2}$$

Here $(f_{0i}, f_{1i}, g_{0i}, g_{1i})$ depend on a variable t:

$$(f_{0i}, f_{1i}, g_{0i}, g_{1i}) = (f_{0i}(t), f_{1i}(t), g_{0i}(t), g_{1i}(t)) \quad (i = 1, \ldots, 8).$$

We then define the following points:

$$\begin{aligned}(\bar{f}_{0i}, \bar{f}_{1i}, \bar{g}_{0i}, \bar{g}_{1i}) &= (f_{0i}(\bar{t}), f_{1i}(\bar{t}), g_{0i}(\bar{t}), g_{1i}(\bar{t})) \quad (i = 1, \ldots, 8),\\ (\underline{f_{0i}}, \underline{f_{1i}}, \underline{g_{0i}}, \underline{g_{1i}}) &= (f_{0i}(\underline{t}), f_{1i}(\underline{t}), g_{0i}(\underline{t}), g_{1i}(\underline{t})) \quad (i = 1, \ldots, 8),\end{aligned}$$

where $\bar{t} = t + \lambda$, $\underline{t} = t - \lambda$, $\lambda = \frac{1}{2}\sum_{i=1}^{8} b_i$.

The $A_0^{(1)}$-surface discrete Painlevé equation ($dP(A_0^{(1)})$) has the following trivial solution moving on the elliptic curve (2.1) [5]:

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = \left(1,\ \wp(q + 2t^2/\lambda + t),\ 1,\ \wp(t - q - 2t^2/\lambda)\right), \tag{2.3}$$

where q is a constant determined by the initial condition. We define the following points:

$$\begin{aligned}(\bar{f}_{0\mathrm{c}}, \bar{f}_{1\mathrm{c}}, \bar{g}_{0\mathrm{c}}, \bar{g}_{1\mathrm{c}}) &= (f_{0\mathrm{c}}(\bar{t}), f_{1\mathrm{c}}(\bar{t}), g_{0\mathrm{c}}(\bar{t}), g_{1\mathrm{c}}(\bar{t})),\\ (\underline{f_{0\mathrm{c}}}, \underline{f_{1\mathrm{c}}}, \underline{g_{0\mathrm{c}}}, \underline{g_{1\mathrm{c}}}) &= (f_{0\mathrm{c}}(\underline{t}), f_{1\mathrm{c}}(\underline{t}), g_{0\mathrm{c}}(\underline{t}), g_{1\mathrm{c}}(\underline{t})).\end{aligned}$$

We now introduce the vectors

$$
\begin{aligned}
&w_{4,1}(f_0,f_1,g_0,g_1)\\
&\qquad = {}^t\left({f_1}^4g_1 \;\; f_0{f_1}^3g_1 \;\; {f_0}^2{f_1}^2g_1 \;\; {f_0}^3f_1g_1 \;\; {f_0}^4g_1 \;\; {f_1}^4g_0 \;\; f_0{f_1}^3g_0 \;\; {f_0}^2{f_1}^2g_0 \;\; {f_0}^3f_1g_0 \;\; {f_0}^4g_0 \right),\\
&v = w_{4,1}(f_0,f_1,g_0,g_1), \qquad \check{v} = w_{4,1}(f_0,f_1,\bar{g}_0,\bar{g}_1),\\
&u = w_{4,1}(g_0,g_1,f_0,f_1), \qquad \hat{u} = w_{4,1}(g_0,g_1,\underline{f_0},\underline{f_1}),\\
&v_i = w_{4,1}(f_{0i},f_{1i},g_{0i},g_{1i}), \qquad \check{v}_i = w_{4,1}(f_{0i},f_{1i},\bar{g}_{0i},\bar{g}_{1i}),\\
&u_i = w_{4,1}(g_{0i},g_{1i},f_{0i},f_{1i}), \qquad \hat{u}_i = w_{4,1}(g_{0i},g_{1i},\underline{f_{0i}},\underline{f_{1i}}) \qquad (i=1,\ldots,8),\\
&v_{\mathrm{c}} = w_{4,1}(f_{0\mathrm{c}},f_{1\mathrm{c}},g_{0\mathrm{c}},g_{1\mathrm{c}}), \qquad \check{v}_{\mathrm{c}} = w_{4,1}(f_{0\mathrm{c}},f_{1\mathrm{c}},\bar{g}_{0\mathrm{c}},\bar{g}_{1\mathrm{c}}),\\
&u_{\mathrm{c}} = w_{4,1}(g_{0\mathrm{c}},g_{1\mathrm{c}},f_{0\mathrm{c}},f_{1\mathrm{c}}), \qquad \hat{u}_{\mathrm{c}} = w_{4,1}(g_{0\mathrm{c}},g_{1\mathrm{c}},\underline{f_{0\mathrm{c}}},\underline{f_{1\mathrm{c}}}).
\end{aligned}
$$

$$
\begin{aligned}
&w_{3,1}(f_0,f_1,g_0,g_1)\\
&\qquad = {}^t\left({f_1}^3g_1 \quad f_0{f_1}^2g_1 \quad {f_0}^2f_1g_1 \quad {f_0}^3g_1 \quad {f_1}^3g_0 \quad f_0{f_1}^2g_0 \quad {f_0}^2f_1g_0 \quad {f_0}^3g_0 \right),\\
&\phi_i = w_{3,1}(f_{0i},f_{1i},g_{0i},g_{1i}), \qquad \check{\phi}_i = w_{3,1}(f_{0i},f_{1i},\bar{g}_{0i},\bar{g}_{1i}),\\
&\psi_i = w_{3,1}(g_{0i},g_{1i},f_{0i},f_{1i}), \qquad \hat{\psi}_i = w_{3,1}(g_{0i},g_{1i},\underline{f_{0i}},\underline{f_{1i}}) \qquad (i=1,\ldots,8).
\end{aligned}
$$

We describe the following theorem by using these vectors.

Theorem 1. *$dP(A_0^{(1)})$ can be written as*

$$
\begin{aligned}
&\det\left(v,v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_{\mathrm{c}}\right)\det\left(\check{v},\check{v}_1,\check{v}_2,\check{v}_3,\check{v}_4,\check{v}_5,\check{v}_6,\check{v}_7,\check{v}_8,\check{v}_{\mathrm{c}}\right)\\
&= \det\left(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8\right)\det\left(\check{\phi}_1,\check{\phi}_2,\check{\phi}_3,\check{\phi}_4,\check{\phi}_5,\check{\phi}_6,\check{\phi}_7,\check{\phi}_8\right)\\
&\quad \times \prod_{i=1}^{8}\left(f_{0\mathrm{c}}f_{1i}-f_{1\mathrm{c}}f_{0i}\right) \times \left(g_{0\mathrm{c}}g_1-g_{1\mathrm{c}}g_0\right)\left(\bar{g}_{0\mathrm{c}}\bar{g}_1-\bar{g}_{1\mathrm{c}}\bar{g}_0\right)\prod_{i=1}^{8}\left(f_{0i}f_1-f_{1i}f_0\right),
\end{aligned}
\tag{2.4a}
$$

$$
\begin{aligned}
&\det\left(u,u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8,u_{\mathrm{c}}\right)\det\left(\hat{u},\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4,\hat{u}_5,\hat{u}_6,\hat{u}_7,\hat{u}_8,\hat{u}_{\mathrm{c}}\right)\\
&= \det\left(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8\right)\det\left(\hat{\psi}_1,\hat{\psi}_2,\hat{\psi}_3,\hat{\psi}_4,\hat{\psi}_5,\hat{\psi}_6,\hat{\psi}_7,\hat{\psi}_8\right)\\
&\quad \times \prod_{i=1}^{8}\left(g_{0\mathrm{c}}g_{1i}-g_{1\mathrm{c}}g_{0i}\right) \times \left(f_{0\mathrm{c}}f_1-f_{1\mathrm{c}}f_0\right)\left(\underline{f_{0\mathrm{c}}}\,\underline{f_1}-\underline{f_{1\mathrm{c}}}\,\underline{f_0}\right)\prod_{i=1}^{8}\left(g_{0i}g_1-g_{1i}g_0\right).
\end{aligned}
\tag{2.4b}
$$

For later use, we calculate the determinants C and D given by:

$$C = \det\left(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8\right)\det\left(\check{\phi}_1,\check{\phi}_2,\check{\phi}_3,\check{\phi}_4,\check{\phi}_5,\check{\phi}_6,\check{\phi}_7,\check{\phi}_8\right) \times \prod_{i=1}^{8}\left(f_{0\mathrm{c}}f_{1i} - f_{1\mathrm{c}}f_{0i}\right), \tag{2.5a}$$

$$D = \det\left(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8\right)\det\left(\hat{\psi}_1,\hat{\psi}_2,\hat{\psi}_3,\hat{\psi}_4,\hat{\psi}_5,\hat{\psi}_6,\hat{\psi}_7,\hat{\psi}_8\right) \times \prod_{i=1}^{8}\left(g_{0\mathrm{c}}g_{1i} - g_{1\mathrm{c}}g_{0i}\right), \tag{2.5b}$$

for which we find:

$$C = \frac{\sigma(4t)^4\sigma(4t+2\lambda)^4\prod_{\substack{i,j=1,\dots,8\\ i<j}}\sigma(b_i-b_j)^2}{\sigma(q+2t^2/\lambda+t)^{16}} \times \prod_{i=1}^{8}\frac{\sigma(q+2t^2/\lambda-b_i)\sigma(q+2t^2/\lambda+b_i+2t)}{\sigma(b_i+t)^{14}\sigma(t-b_i)^2\sigma(t+\lambda-b_i)^2}, \tag{2.6a}$$

$$D = \frac{\sigma(4t)^4\sigma(4t-2\lambda)^4\prod_{\substack{i,j=1,\dots,8\\ i<j}}\sigma(b_i-b_j)^2}{\sigma(t-q-2t^2/\lambda)^{16}} \times \prod_{i=1}^{8}\frac{\sigma(q+2t^2/\lambda-b_i)\sigma(2t-q-2t^2/\lambda-b_i)}{\sigma(t-b_i)^{14}\sigma(b_i+t)^2\sigma(b_i+t-\lambda)^2}. \tag{2.6b}$$

Remark 2.1. We can parametrize an isomorphism class of surfaces by using the period mapping. The period mapping maps the elements of the second homology to $\mathbb{C}$.

Let ω be a meromorphic 2-form on X with $\mathrm{div}(\omega) = -D$. Then ω determines a period mapping $\hat{\chi}\colon H_2(X-D,\mathbb{Z}) \to \mathbb{C}$ which sends $\Gamma \in H_2(X-D,\mathbb{Z})$ to $\int_\Gamma \omega$.

Now, there exists a short exact sequence:

$$0 \to H_1(D,\mathbb{Z}) \to H_2(X-D,\mathbb{Z}) \to Q(E_8^{(1)}) \to 0,$$

where $Q(E_8^{(1)}) = \sum_{i=0}^{8}\mathbb{Z}\alpha_i$ is the root lattice of type $E_8^{(1)}$. So we obtain the mapping

$$\chi\colon Q(E_8^{(1)}) \to \mathbb{C} \qquad \mathrm{mod}\ \ \hat{\chi}(H_1(D,\mathbb{Z}))$$

through the period mapping $\hat{\chi}$. In this case, the parametrization is

$$\begin{gathered}\chi(\alpha_1) = -4t, \quad \chi(\alpha_2) = b_1 + b_2 + 2t, \quad \chi(\alpha_i) = b_i - b_{i-1} \ (i = 3, \dots, 7), \\ \chi(\alpha_8) = b_2 - b_1, \quad \chi(\alpha_0) = b_8 - b_7.\end{gathered} \tag{2.7}$$

Here $Q(E_8^{(1)})$ is realized in $\mathrm{Pic}(X) = H_2(X, \mathbb{Z})$. And α_i's are represented by elements of the Picard group as follows:

$$\begin{gathered}\alpha_1 = H_1 - H_0, \quad \alpha_2 = H_0 - E_1 - E_2, \quad \alpha_i = E_{i-1} - E_i \ (i = 3, \dots, 7), \\ \alpha_8 = E_1 - E_2, \quad \alpha_0 = E_7 - E_8.\end{gathered} \tag{2.8}$$

We denote the total transform of $f_1 = \text{constant} \times f_0$, (or $g_1 = \text{constant} \times g_0$) on X by H_0 (or H_1 respectively) and the total transform of the point p_i by E_i. The Picard group $\mathrm{Pic}(X)$ and canonical divisor $\mathcal{K}_X$ are

$$\mathrm{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \sum_{i=1}^{8} \mathbb{Z}E_i, \quad \mathcal{K}_X = -2H_0 - 2H_1 + \sum_{i=1}^{8} E_i,$$

where the intersection numbers of pairs of base elements are

$$H_i \cdot H_j = 1 - \delta_{i,j}, \quad E_i \cdot E_j = -\delta_{i,j}, \quad H_i \cdot E_j = 0, \quad \text{where } \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

The generators of the affine Weyl group $W(E_8^{(1)}) = \langle w_i \ (i = 0, 1, \dots, 8) \rangle$ will act on the total transforms. We give a representation of these actions that will enable us to construct $dP(A_0^{(1)})$.

$$\begin{aligned}
&w_1 \colon (H_0, H_1) \mapsto (H_1, H_0), \\
&w_2 \colon (H_1, E_1, E_2) \mapsto (H_1 + H_0 - E_1 - E_2, H_0 - E_2, H_0 - E_1), \\
&w_i \colon (E_{i-1}, E_i) \mapsto (E_i, E_{i-1}) \quad (i = 3, \dots, 7), \\
&w_8 \colon (E_1, E_2) \mapsto (E_2, E_1), \\
&w_0 \colon (E_7, E_8) \mapsto (E_8, E_7).
\end{aligned}$$

By taking a translation contained in $W(E_8^{(1)})$, we obtain a nonlinear difference equation. The translation can be described by a product of simple reflections w_i:

$$\begin{gathered}dP(A_0^{(1)}) = w_1 \circ r^{-1} \circ w_1 \circ r \colon \\ (b_i, t, f_0 : f_1, g_0 : g_1) \mapsto (b_i, t + \lambda, \bar{f}_0 : \bar{f}_1, \bar{g}_0 : \bar{g}_1) \quad (i = 1, \dots, 8), \\ \lambda = \frac{1}{2} \sum_{i=1}^{8} b_i,\end{gathered} \tag{2.9}$$

where

$$r = w_2 \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_7 \circ w_0 \circ w_8 \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_7 \circ w_2 \circ \\ \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_8 \circ w_3 \circ w_4 \circ w_5 \circ w_2 \circ w_3 \circ w_4 \circ w_8 \circ w_3 \circ w_2. \tag{2.10}$$

An element r of the affine Weyl group acts on these total transforms as

$$r\colon (H_0, H_1, E_i) \mapsto (H_0, H_1 + 4H_0 - \sum_{i=1}^{8} E_i, H_0 - E_{9-i}). \tag{2.11}$$

Equation (2.4a) describes exactly this action. The following gives a presentation of the action of r on coordinates and parameters:

$$\begin{aligned} r\colon (f_0 : f_1,\, g_0 : g_1, f_{0i} : f_{1i},\, g_{0i} : g_{1i}) & \\ \mapsto (\check{f}_0 : \check{f}_1,\, \check{g}_0 : \check{g}_1, \check{f}_{0i} : \check{f}_{1i},\, \check{g}_{0i} : \check{g}_{1i}) & \\ = (f_0 : f_1,\, \bar{g}_0 : \bar{g}_1, f_{0(9-i)} : f_{1(9-i)},\, \bar{g}_{0(9-i)} : \bar{g}_{1(9-i)}). & \end{aligned} \tag{2.12}$$

For example, if we impose $\check{g}_{0\mathrm{c}}\check{g}_1 - \check{g}_{1\mathrm{c}}\check{g}_0 = 0$ in the equation (2.4a), i.e. we impose $\bar{g}_{0\mathrm{c}}\bar{g}_1 - \bar{g}_{1\mathrm{c}}\bar{g}_0 = 0$ on $(f_0 : f_1,\, \bar{g}_0 : \bar{g}_1)$ in the equation, we obtain $\det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_{\mathrm{c}}) = 0$ on $(f_0 : f_1,\, g_0 : g_1)$ as an equivalent condition. This expresses the action $r\colon H_1 \mapsto H_1 + 4H_0 - \sum_{i=1}^{8} E_i$. If we impose $\check{f}_{0i}\check{f}_1 - \check{f}_{1i}\check{f}_0 = 0$, $\check{g}_{0i}\check{g}_1 - \check{g}_{1i}\check{g}_0 = 0$ in the equation (2.4a), i.e. we impose $f_{0(9-i)}f_1 - f_{1(9-i)}f_0 = 0$, $\bar{g}_{0(9-i)}\bar{g}_1 - \bar{g}_{1(9-i)}\bar{g}_0 = 0$ on $(f_0 : f_1,\, \bar{g}_0 : \bar{g}_1)$ in the equation, we obtain $f_{0(9-i)}f_1 - f_{1(9-i)}f_0 = 0$ on $(f_0 : f_1,\, g_0 : g_1)$. This expresses $r\colon E_i \mapsto H_0 - E_{9-i}$.

If we decide the form of the equation in this way, an undetermined constant will remain. However the condition that the trivial solution (2.3) (for arbitrary q) satisfies the equation completely determines the form of (2.4).

Remark 2.2. We presented the $dP(A_0^{(1)})$ in [5] as follows. That is, in this section we have rewritten this expression. This system is equivalent to ell.P derived in [8]. The following 2×2 matrices represent PGL(2)-action on $\mathbb{P}^1$, i.e., $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z$ means $w = (az + b)/(cz + d)$. The $A_0^{(1)}$-surface discrete Painlevé equation is the following difference system for unknown functions

$f(t)$, $g(t)$:

$$\begin{aligned}\bar{g} = M&\left(f, c_7, c_8, t - \frac{1}{4}\sum_{i=1}^{6} c_i\right) M\left(f, c_5, c_6, t - \frac{1}{4}\sum_{i=1}^{4} c_i\right)\\ &\times M\left(f, c_3, c_4, t - \frac{1}{4}(c_1 + c_2)\right) M(f, c_1, c_2, t)\, g,\end{aligned} \tag{2.13a}$$

$$\begin{aligned}\underline{f} = M&\left(g, d_7, d_8, t - \frac{1}{4}\sum_{i=1}^{6} d_i\right) M\left(g, d_5, d_6, t - \frac{1}{4}\sum_{i=1}^{4} d_i\right)\\ &\times M\left(g, d_3, d_4, t - \frac{1}{4}(d_1 + d_2)\right) M(g, d_1, d_2, t)\, f,\end{aligned} \tag{2.13b}$$

where $\bar{g} = g(t+\lambda)$, $\underline{f} = f(t-\lambda)$ and

$$\begin{aligned}&M(h, \kappa_1, \kappa_2, s)\\ &= \begin{pmatrix} -\wp(2s - \frac{-\kappa_1+\kappa_2}{2}) & \wp(2s - \frac{\kappa_1-\kappa_2}{2}) \\ -1 & 1 \end{pmatrix}\\ &\times \begin{pmatrix} (h-\wp(\kappa_2))(\wp(2s)-\wp(2s-\kappa_2))(\wp(2s-\frac{\kappa_1+\kappa_2}{2})-\wp(2s-\frac{\kappa_1-\kappa_2}{2})) & 0 \\ 0 & (h-\wp(\kappa_1))(\wp(2s)-\wp(2s-\kappa_1))(\wp(2s-\frac{\kappa_1+\kappa_2}{2})-\wp(2s-\frac{-\kappa_1+\kappa_2}{2})) \end{pmatrix}\\ &\times \begin{pmatrix} 1 & -\wp(2s-\kappa_1) \\ 1 & -\wp(2s-\kappa_2) \end{pmatrix}.\end{aligned} \tag{2.14}$$

Here b_i $(i = 1, \ldots, 8)$ are constant parameters and we set $\lambda = \frac{1}{2}\sum_{i=1}^{8} b_i, c_i = b_i + t, d_i = t - b_i$. Note that we will regard $(f(t), g(t))$ as inhomogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$.

3 $A_0^{(1)*}$-surface

We construct the $A_0^{(1)*}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points. These eight points and a curve through these points are

$$f_1{}^2 g_0{}^2 + f_0{}^2 g_1{}^2 - \left(t^2 + \frac{1}{t^2}\right) f_0 f_1 g_0 g_1 + \left(t^2 - \frac{1}{t^2}\right)^2 f_0{}^2 g_0{}^2 = 0, \tag{3.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = \left(1 : b_i t + \frac{1}{b_i t}, 1 : \frac{t}{b_i} + \frac{b_i}{t}\right) \quad (i = 1, \ldots, 8). \tag{3.2}$$

The $A_0^{(1)*}$-surface discrete Painlevé equation $(dP(A_0^{(1)*}))$ has the following

trivial solution moving on the curve (3.1) [5]:

$$
\begin{aligned}
&(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) \\
&= \left(1, tq\exp\left(\frac{2(\log t)^2}{\log\lambda}\right) + \frac{1}{tq\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)}, 1, \frac{t}{q\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)} + \frac{q\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)}{t}\right),
\end{aligned}
\tag{3.3}
$$

where q is a constant determined by the initial condition.

Using these expressions we formulate the theorem:

Theorem 2. *$dP(A_0^{(1)*})$ can be written in the form (2.4).*

Let $q = 0$, then the trivial solution (3.3) is

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0, 1, 0, 1)\,, \tag{3.4}$$

and C, D (2.5) are

$$C = \left(t^2 - \frac{1}{t^2}\right)^4 \left(t\overline{t} - \frac{1}{t\overline{t}}\right)^4 \prod_{\substack{i,j=1,\dots,8\\ i<j}} (b_i - b_j)^2 \Bigg/ \prod_{i=1}^{8} {b_i}^7, \tag{3.5a}$$

$$D = \left(t^2 - \frac{1}{t^2}\right)^4 \left(t\underline{t} - \frac{1}{t\underline{t}}\right)^4 \prod_{\substack{i,j=1,\dots,8\\ i<j}} (b_i - b_j)^2 \Bigg/ \prod_{i=1}^{8} {b_i}^7. \tag{3.5b}$$

4 $A_0^{(1)**}$-surface

We construct the $A_0^{(1)**}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points. These eight points and a curve on which these points lie are as follows:

$$(f_1 g_0 - f_0 g_1)^2 - 8t^2(f_0 f_1 {g_0}^2 + {f_0}^2 g_0 g_1) + 16t^4 {f_0}^2 {g_0}^2 = 0, \tag{4.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\, g_{0i} : g_{1i}) = \left(1 : (b_i + t)^2, 1 : (t - b_i)^2\right) \quad (i = 1, \dots, 8), \tag{4.2}$$

The $A_0^{(1)**}$-surface discrete Painlevé equation ($dP(A_0^{(1)**})$) has the following trivial solution moving on the curve (4.1) [5]:

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = \left(1,\ \left(q + 2t^2/\lambda + t\right)^2,\ 1,\ \left(t - q - 2t^2/\lambda\right)^2\right), \tag{4.3}$$

where q is a constant determined by initial condition.

Theorem 3. *$dP(A_0^{(1)**})$ can be written in the form (2.4).*

If $q=\infty$ the trivial solution (4.3) is

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0,1,0,1) \tag{4.4}$$

and C, D (2.5) are

$$C = 4096\, t^4 \,(t+\bar{t})^4 \prod_{\substack{i,j=1,\ldots,8\\ i<j}} (b_i-b_j)^2\,, \tag{4.5a}$$

$$D = 4096\, t^4 \,(t+\underline{t})^4 \prod_{\substack{i,j=1,\ldots,8\\ i<j}} (b_i-b_j)^2\,. \tag{4.5b}$$

5 $A_1^{(1)}$-surface

We construct the $A_1^{(1)}$-surface by blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at eight points. These eight points and a curve through these points are

$$(f_1g_1 - t^2 f_0 g_0)(f_1 g_1 - f_0 g_0) = 0, \tag{5.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = \begin{cases} \left(1 : b_i t,\ 1 : \frac{t}{b_i}\right) & (i=1,\ldots,4),\\ \left(1 : b_i,\ 1 : \frac{1}{b_i}\right) & (i=5,\ldots,8). \end{cases} \tag{5.2}$$

The $A_1^{(1)}$-surface discrete Painlevé equation ($dP(A_1^{(1)})$) has the following trivial solution.

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0,\ 1,\ 1,\ 0)\,.$$

Theorem 4. *$dP(A_1^{(1)})$ can be written in the form (2.4).*

C, D (2.5) are

$$C = t^8\left(1-t^2\right)^4 (1-t\bar{t})^4 \prod_{\substack{i,j=1,\ldots,4\\ i<j}} (b_i-b_j)^2 \prod_{\substack{i,j=5,\ldots,8\\ i<j}} (b_i-b_j)^2 \Bigg/ \prod_{i=1}^{8} b_i, \tag{5.3a}$$

$$D = t^{12}\left(1-t^2\right)^4 (1-t\underline{t})^4 \prod_{\substack{i,j=1,\ldots,4\\ i<j}} (b_i-b_j)^2 \prod_{\substack{i,j=5,\ldots,8\\ i<j}} (b_i-b_j)^2 \Bigg/ \prod_{i=1}^{8} {b_i}^6. \tag{5.3b}$$

6 $A_1^{(1)*}$-surface

We construct the $A_1^{(1)*}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points. These eight points and a curve passing through these points are

$$(f_1 g_0 + f_0 g_1 - 2t f_0 g_0)(f_1 g_0 + f_0 g_1) = 0, \tag{6.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = \begin{cases} (1 : b_i + t,\ 1 : t - b_i) & (i = 1, \ldots, 4), \\ (1 : b_i,\ 1 : -b_i) & (i = 5, \ldots, 8). \end{cases} \tag{6.2}$$

The $A_1^{(1)*}$-surface discrete Painlevé equation ($dP(A_1^{(1)*})$) has the following trivial solution.

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0,\ 1,\ 0,\ 1)$$

and we have the following theorem:

Theorem 5. *$dP(A_1^{(1)*})$ can be written in the form (2.4).*

C, D (2.5) are

$$C = 16t^4 \left(t + \overline{t}\right)^4 \prod_{\substack{i,j=1,\ldots,4 \\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\ldots,8 \\ i<j}} (b_i - b_j)^2\,, \tag{6.3a}$$

$$D = 16t^4 \left(t + \underline{t}\right)^4 \prod_{\substack{i,j=1,\ldots,4 \\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\ldots,8 \\ i<j}} (b_i - b_j)^2\,. \tag{6.3b}$$

7 $A_2^{(1)}$-surface

We construct $A_2^{(1)}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points. These eight points and a curve on which these points lie are

$$f_0 g_0 (f_1 g_1 - f_0 g_0) = 0, \tag{7.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = \begin{cases} (1 : b_i t,\ 0 : 1) & (i = 1, 2), \\ \left(0 : 1,\ 1 : \frac{t}{b_i}\right) & (i = 3, 4), \\ \left(1 : b_i,\ 1 : \frac{1}{b_i}\right) & (i = 5, \ldots, 8). \end{cases} \tag{7.2}$$

The $A_2^{(1)}$-surface discrete Painlevé equation ($dP(A_2^{(1)})$) has the trivial solution

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0,\ 1,\ 1,\ 0)\,.$$

If we write $dP(A_2^{(1)})$ in the form (2.4), then both sides of (2.4a) are 0. However we can derive the following expression from the form of $dP(A_1^{(1)})$.

Theorem 6. *$dP(A_2^{(1)})$ can be written as*

$$\begin{aligned}
&\det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \mathrm{v_c}) \det(\check{v}, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{\mathrm{v}}_\mathrm{c}) \\
&\qquad = t^3 \bar{t}^3 (b_1 - b_2)^2 (b_3 - b_4)^2 \prod_{\substack{i,j=5,\dots,8 \\ i<j}} (b_i - b_j)^2 \Big/ {b_3}^3 {b_4}^3 \\
&\qquad\quad \times (g_{0\mathrm{c}} g_1 - g_{1\mathrm{c}} g_0)(\bar{g}_{0\mathrm{c}} \bar{g}_1 - \bar{g}_{1\mathrm{c}} \bar{g}_0) \prod_{i=1}^{8} (f_{0i} f_1 - f_{1i} f_0) ,
\end{aligned} \tag{7.3a}$$

$$\begin{aligned}
&\det(u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_\mathrm{c}) \det(\hat{u}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8, \hat{u}_\mathrm{c}) \\
&\qquad = t^9 \underline{t} (b_1 - b_2)^2 (b_3 - b_4)^2 \prod_{\substack{i,j=5,\dots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=3}^{8} {b_i}^5 \\
&\qquad\quad \times (f_{0\mathrm{c}} f_1 - f_{1\mathrm{c}} f_0) \left(\underline{f_{0\mathrm{c}}}\, \underline{f_1} - \underline{f_{1\mathrm{c}}}\, \underline{f_0} \right) \prod_{i=1}^{8} (g_{0i} g_1 - g_{1i} g_0) ,
\end{aligned} \tag{7.3b}$$

where

$$\mathrm{v_c} = \check{\mathrm{v}}_\mathrm{c} = {}^t \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

8 $A_3^{(1)}$-surface

We construct the $A_3^{(1)}$-surface by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points. These eight points and a curve through them are

$$f_0 f_1 g_0 g_1 = 0, \tag{8.1}$$

$$p_i\colon\ (f_{0i} : f_{1i},\ g_{0i} : g_{1i}) = \begin{cases} (1 : b_i, 0 : 1) & (i = 1, 2), \\ \left(0 : 1, 1 : \frac{1}{b_i}\right) & (i = 3, 4), \\ (1 : b_i t, 1 : 0) & (i = 5, 6), \\ \left(1 : 0, 1 : \frac{t}{b_i}\right) & (i = 7, 8). \end{cases} \tag{8.2}$$

The $A_3^{(1)}$-surface discrete Painlevé equation ($dP(A_3^{(1)})$) has the trivial solution.

$$(f_{0\mathrm{c}}, f_{1\mathrm{c}}, g_{0\mathrm{c}}, g_{1\mathrm{c}}) = (0,\ 1,\ 1,\ 0) ,$$

for which we have

Theorem 7. *$dP(A_3^{(1)})$ can be written as*

$$\det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \mathrm{v_c}) \det(\check{v}, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{\mathrm{v}}_\mathrm{c})$$
$$= \frac{1}{(b_1-b_2)^2(b_3-b_4)^2(b_5-b_6)^2(b_7-b_8)^2 t^7 \bar{t}} \left(\frac{b_3b_4b_7b_8}{b_1b_2b_5b_6}\right)^2 \tag{8.3a}$$
$$\times\, g_1\bar{g}_1(f_1-b_1f_0)(f_1-b_2f_0){f_0}^2(f_1-b_5tf_0)(f_1-b_6tf_0){f_1}^2,$$
$$\det(u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, \mathrm{u_c}) \det(\hat{u}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8, \hat{\mathrm{u}}_\mathrm{c})$$
$$= \frac{{b_3}^4{b_4}^4{b_7}^6{b_8}^6}{(b_1-b_2)^2(b_3-b_4)^2(b_5-b_6)^2(b_7-b_8)^2 t^{11} \bar{t}} \tag{8.3b}$$
$$\times\, f_0\underline{f_0}{g_0}^2(g_1-1/b_3g_0)(g_1-1/b_4g_0){g_1}^2(g_1-t/b_7g_0)(g_1-t/b_8g_0),$$

where

$$\mathrm{u_c} = \hat{\mathrm{u}}_\mathrm{c} = {}^t\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}.$$

If we expand the determinant in (8.3), we obtain

$$\frac{g_1\bar{g}_1}{g_0\bar{g}_0} = \frac{(f_1-b_5tf_0)(f_1-b_6tf_0)}{(f_1-b_1f_0)(f_1-b_2f_0)}, \tag{8.4a}$$
$$\frac{f_1\underline{f_1}}{f_0\underline{f_0}} = \frac{(g_1-t/b_7g_0)(g_1-t/b_8g_0)}{(g_1-1/b_3g_0)(g_1-1/b_4g_0)}. \tag{8.4b}$$

This is exactly q-$\mathrm{P_{VI}}$ [3].

9 Discussion

In this paper we presented a new representation of discrete Painlevé equations. Up to now, the complexity of some of the difference Painlevé equations prevented us from studying their properties. It is our hope that these new forms of these equations, especially for $dP(A_0^{(1)})$, $dP(A_0^{(1)*})$, and $dP(A_0^{(1)**})$ which have Weyl groups of type $E_8^{(1)}$, will prove useful in such analysis.

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References

[1] B. Grammaticos, A. Ramani and V. G. Papageorgiou, Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1991), 1825–1828.

[2] B. Grammaticos, Y. Ohta, A. Ramani and H. Sakai, Degeneration through coalescence of the q-Painlevé VI equations, *J. Phys. A: Math. Gen.* **31** (1998), 3545–3558.

[3] M. Jimbo and H. Sakai, A q-analog of the sixth Painlevé equation, *Lett. Math. Phys.* **38** (1996), 145–154.

[4] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, ${}_{10}E_9$ solution to the elliptic Painlevé equation, *J. Phys. A: Math. Gen.* **36** (2003), 263–272.

[5] M. Murata, H. Sakai and J. Yoneda, Riccati Solutions of Discrete Painlevé Equations with Weyl Group Symmetry of Type $E_8^{(1)}$, *J. Math. Phys.* **44** (2003), 1396–1414.

[6] Y. Ohta, A. Ramani and B. Grammaticos, An affine Weyl group approach to the 8-parameter discrete Painlevé equation, *J. Phys. A: Math. Gen.* **34** (2001), 10523–10532.

[7] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* **67** (1991), 1829–1832.

[8] H. Sakai, Rational Surfaces Associated with Affine Root Systems and Geometry of the Painlevé Equations, *Commun. Math. Phys.* **220** (2001), 165–229.